Infinitesimal Deformation of Surfaces.

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I.

Consider a flexible and inextensible surface S referred to a general system of parametric lines, u and v, and upon it a point M whose cartesian coordinates are x, y, z. Effect upon S a deformation which changes it into a surface S' and in such a way that M goes into a point M' whose coordinates are given by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \varepsilon z_1,$$

where x_1 y_1, z_1 are functions of u and v and ε is a small constant whose powers higher than the first are neglected. Since S is inextensible, $ds' = ds$, so that the functions x_1, y_1, z_1 must satisfy the condition

$$dx dx_1 + dy dy_1 + dz dz_1 = 0. \qquad (\alpha)$$

Hence the effecting of the required deformation depends upon the integration of this total differential equation, for as soon as a solution is obtained an infinitesimal deformation can be made, in directions whose cosines are proportional to x_1, y_1, z_1.

As Moutard has pointed out, the functions x_1, y_1, z_1 may be looked upon as the cartesian coordinates of a surface S_1, and in consequence of condition (α), S_1 and S correspond by orthogonality of linear elements.

In section II, we determine the coefficients of the two fundamental quadratic forms of S_1 as functions of the coefficients of the quadratic forms of S and the function, to be defined later, which determines the character of the given deformation. These enable us to determine the correspondence between the conjugate and asymptotic lines on the two surfaces, and the relative character of the total curvature of these surfaces at corresponding points. We also find the relations between the Christoffel symbols formed with respect to the square of linear element of S and S_1, and from these learn the properties of the partial

differential equations, which the cartesian coordinates of a point on these surfaces satisfy when the latter are referred to particular systems of corresponding lines. And the special case is considered when the tangent planes to S and S_1 are parallel.

In section III, the surface S', which is the result of the deformation of S is studied by means of the coefficients of its two fundamental forms which are calculated as functions of the coefficients of the fundamental forms of S and S_1, and the characteristic function of the deformation. From the coefficients of the first form we pass to the Christoffel symbols, formed with respect to this form, and find the differential equations which the coordinates of S' satisfy, and by means of the coefficients of the second form, we are enabled to discover the correspondence between conjugate lines and asymptotic lines on S, S_1 and S'. Finally, we find the necessary and sufficient condition that S has been displaced merely and not deformed.

In the last section, several kinds of infinitesimal deformation of a minimal surface are treated, and a relation, from the point of view of infinitesimal deformation, is found to exist between a minimal surface, its associates and adjunct. After the consideration of the deformation of a sphere, the section closes with a discussion of two particular kinds of deformation: 1° where S and S_1 are deformed into S' and S_1', the latter corresponding by orthogonality of linear elements; 2° the case where S and S_0, the latter being a surface corresponding to S by parallelism of tangent planes, are so deformed that the resulting surfaces have the same kind of correspondence.

II

Throughout this discussion we shall use the following notation:

$$\begin{aligned}
&E=\sum\left(\frac{\partial x}{\partial u}\right)^2, \quad F=\sum\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}, \quad G=\sum\left(\frac{\partial x}{\partial v}\right)^2;\\
&D=\sum X\frac{\partial^2 x}{\partial u^2}, \quad D'=\sum X\frac{\partial^2 x}{\partial u\,\partial v}, \quad D''=\sum X\frac{\partial^2 x}{\partial v^2};\\
&\mathfrak{E}=\sum\left(\frac{\partial X}{\partial u}\right)^2, \quad \mathfrak{F}=\sum\frac{\partial X}{\partial u}\,\frac{\partial X}{\partial v}, \quad \mathfrak{G}=\sum\left(\frac{\partial X}{\partial v}\right)^2,
\end{aligned} \tag{A}$$

where X, Y, Z are the direction cosines of the normal to the surface. If ds and $d\sigma$ denote the elements of arc on the surface S and its spherical representa-

tion respectively, then

$$\begin{aligned} ds^2 &= E du^2 + 2F du\, dv + G dv^2, \\ d\sigma^2 &= \mathfrak{E} du^2 + 2\mathfrak{F} du\, dv + \mathfrak{G} dv^2, \end{aligned} \tag{B}$$

And for brevity we put

$$H = \sqrt{EG - F^2}, \quad \mathfrak{H} = \sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}, \quad K = \frac{DD'' - D'^2}{H^2}. \tag{C}$$

Let S be referred to a general system of parametric lines and take for lines of reference upon S_1 the corresponding system; then

$$dx\, dx_1 + dy\, dy_1 + dz\, dz_1 = 0 \tag{1}$$

may be replaced by the three equations:

$$\sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial v} = 0, \quad \sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial v} + \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial u} = 0. \tag{2}$$

We shall make use of Weingarten's* function ϕ, defined by the equations

$$\sum \frac{\partial x}{\partial u}\,\frac{\partial x_1}{\partial v} = \phi H, \quad \sum \frac{\partial x}{\partial v}\,\frac{\partial x_1}{\partial u} = -\phi H, \tag{3}$$

and which Bianchi† has called the *characteristic function.* By employing the function thus defined, the differential quotients of x_1 can be expressed in the form‡

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{D\left(\phi \frac{\partial X}{\partial v} - X\frac{\partial \phi}{\partial v}\right) - D'\left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u}\right)}{HK}, \\ \frac{\partial x_1}{\partial v} &= \frac{D'\left(\phi \frac{\partial X}{\partial v} - X\frac{\partial \phi}{\partial v}\right) - D''\left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u}\right)}{HK}, \end{aligned} \tag{4}$$

and analogous expressions in y_1 and z_1. From these forms it is seen that as soon as ϕ is known, a surface S_1, which corresponds to S by orthogonality of linear elements, can be found by quadratures; and, moreover, such function ϕ, if it exists, must satisfy the partial differential equation of the second order:

$$\frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{D''\frac{\partial \phi}{\partial u} - D'\frac{\partial \phi}{\partial v}}{HK}\right) + \frac{\partial}{\partial v}\left(\frac{D\frac{\partial \phi}{\partial v} - D'\frac{\partial \phi}{\partial u}}{HK}\right)\right] = \frac{2FD' - ED'' - GD}{H^2}\,\phi, \tag{5}$$

* Crelle, 100, p. 300. † Lezioni, p. 276. ‡ Crelle, 100, p. 303.

which is known as the *characteristic equation.* Bianchi* shows that the characteristic equation can be put in the form

$$D''\left[\frac{\partial^2\phi}{\partial u^2}-\begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u}-\begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v}+\mathcal{E}\phi\right]$$
$$-2D'\left[\frac{\partial^2\phi}{\partial u\,\partial v}-\begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u}-\begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v}+\mathcal{F}\phi\right]$$
$$+\;D\left[\frac{\partial^2\phi}{\partial v^2}-\begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u}-\begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v}+\mathcal{G}\phi\right]=0, \qquad (5')$$

where the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}'$ are formed with respect to the square of the linear element of the spherical representation of S, and, furthermore, that to every solution ϕ of this equation there corresponds a solution of the problem.

From the form of this equation we see that the differential equation of its characteristics is

$$D du^2 + 2D' du\, dv + D'' dv^2 = 0;$$

hence we have the following theorem of Lecornu:†

The asymptotic lines of a surface are the characteristics of its infinitesimal deformation.

In (4) replace the partial differential quotients of X by their expressions in terms of $\frac{\partial x}{\partial u}, \ldots, \frac{\partial z}{\partial v}$; then

$$\begin{aligned}\frac{\partial x_1}{\partial u}&=\frac{\phi}{H}\left(F\frac{\partial x}{\partial u}-E\frac{\partial x}{\partial v}\right)-\frac{X}{HK}\left(D\frac{\partial\phi}{\partial v}-D'\frac{\partial\phi}{\partial u}\right),\\ \frac{\partial x_1}{\partial v}&=\frac{\phi}{H}\left(G\frac{\partial x}{\partial u}-F\frac{\partial x}{\partial v}\right)-\frac{X}{HK}\left(D'\frac{\partial\phi}{\partial v}-D''\frac{\partial\phi}{\partial u}\right),\end{aligned} \qquad (6)$$

and similar ones in y_1 and z_1.

Denoting by E_1, F_1, G_1 the first fundamental coefficients of S_1, we have, by definition:

$$E_1=\sum\left(\frac{\partial x_1}{\partial u}\right)^2,\quad F_1=\sum\frac{\partial x_1}{\partial u}\,\frac{\partial x_1}{\partial v},\quad G_1=\sum\left(\frac{\partial x_1}{\partial v}\right)^2,$$

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* Bianchi, Lezioni, p. 277.

† "Sur l'équilibre des surfaces flexibles et inextensibles" (Journal de l'Ecole Polytechnique, t. XXIX).

and when $\frac{\partial x_1}{\partial u}$, etc., are replaced by their expressions from (6), we get

$$\left.\begin{aligned} E_1 &= E\phi^2 + \frac{1}{H^2K^2}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)^2, \\ F_1 &= F\phi^2 + \frac{1}{H^2K^2}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right), \\ G_1 &= G\phi^2 + \frac{1}{H^2K^2}\left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right)^2, \end{aligned}\right\} \tag{7}$$

then the square of the linear element of S_1 takes the form

$$ds_1^2 = \phi^2 ds^2 + \frac{1}{H^2K^2}\left[\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)du + \left(D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}\right)dv\right]^2, \tag{8}$$

and this can be written

$$ds_1^2 = \phi^2 ds^2 + \lambda^2 d\theta^2,$$

where λ and θ are functions of u and v, whose forms depend upon the surface S and the nature of the deformation, that is, upon the character of ϕ. Comparing this expression with (8), we see that λ must satisfy the condition

$$\frac{\partial}{\partial v}\left[\frac{D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}}{HK\lambda}\right] = \frac{\partial}{\partial u}\left[\frac{D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}}{HK\lambda}\right]. \tag{9}$$

If these expressions are developed and in the reduction use is made of the characteristic equation (5), it is seen that the function λ must satisfy the partial differential equation

$$\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\frac{\partial\lambda}{\partial v} + \left(D''\frac{\partial\phi}{\partial u} - D'\frac{\partial\phi}{\partial v}\right)\frac{\partial\lambda}{\partial u} + K(ED'' + GD - 2FD')\,\phi\lambda = 0, \tag{10}$$

which is seen to involve the fundamental coefficients of S and the characteristic function ϕ. Having obtained a solution of this equation, the corresponding value of θ can be found by quadratures, since

$$\frac{\partial\theta}{\partial u} = \frac{D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}}{HK\lambda}, \qquad \frac{\partial\theta}{\partial v} = \frac{D'\frac{\partial\phi}{\partial v} - D''\frac{\partial\phi}{\partial u}}{HK\lambda}.$$

From the form of the equation, it follows that (10) is satisfied identically when $\phi = 0$. In this case S_1 reduces to a point.

In the special case where S is a minimal surface, the coefficient of λ in (10) vanishes, and hence this equation is satisfied by

$$\lambda = \text{const.}$$

Since the case where this constant is unity is perfectly general, it is clear that the square of the linear element of S_1 can take the form only when S is a minimal surface,

$$ds_1^2 = \phi^2 ds^2 + d\theta^2,$$

Conversely, when the square of the linear element of S_1 can be put in this form S is a minimal surface.

Write
$$A_1 = \frac{\partial (y_1, z_1)}{\partial (u, v)}, \quad B_1 = \frac{\partial (z_1, x_1)}{\partial (u, v)}, \quad C_1 = \frac{\partial (x_1, y_1)}{\partial (u, v)};$$

on introducing, for the sake of brevity, two functions, M and N, defined by

$$M = D\frac{\partial \phi}{\partial v} - D'\frac{\partial \phi}{\partial u}, \quad N = D'\frac{\partial \phi}{\partial v} - D''\frac{\partial \phi}{\partial u}, \tag{11}$$

the above functions can be expressed, by means of (6), in the form

$$\begin{aligned} A_1 &= A\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial x}{\partial v} - N\frac{\partial x}{\partial u}\right), \\ B_1 &= B\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial y}{\partial v} - N\frac{\partial y}{\partial u}\right), \\ C_1 &= C\phi^2 - \frac{\phi}{HK}\left(M\frac{\partial z}{\partial v} - N\frac{\partial z}{\partial u}\right), \end{aligned} \tag{12}$$

and

$$H_1^2 = A_1^2 + B_1^2 + C_1^2 = H^2\phi^4 + \frac{\phi^2}{H^2K^2}(EN^2 - 2FMN + M^2G). \tag{13}$$

For the sake of brevity, write

$$\delta = \pm\sqrt{H^4K^2\phi^2 + (EN^2 - 2FMN + GM^2)} = \frac{H_1HK}{\phi}, \tag{14}$$

then

$$X_1 = \frac{A_1}{H_1} = \frac{HK\phi A - \left(M\frac{\partial x}{\partial v} - N\frac{\partial x}{\partial u}\right)}{\delta}, \tag{15}$$

and similar expressions for Y_1 and Z_1.

From (6) we get

$$\frac{\partial x_1}{\partial u} = \frac{\phi}{H}\left(F\frac{\partial x}{\partial u} - E\frac{\partial x}{\partial v}\right) - \frac{XM}{HK};$$

differentiating with respect to v,

$$\frac{\partial^2 x_1}{\partial u\,\partial v}=\frac{\partial x}{\partial u}\left[F\frac{\partial}{\partial v}\frac{\Phi}{H}+\frac{\Phi}{H}\left(\frac{\partial F}{\partial v}+F\begin{Bmatrix}12\\1\end{Bmatrix}-E\begin{Bmatrix}22\\1\end{Bmatrix}\right)\right]-X\frac{\partial}{\partial v}\frac{M}{\mathcal{H}}$$
$$+\frac{\partial x}{\partial v}\left[-E\frac{\partial}{\partial v}\frac{\Phi}{H}+\frac{\Phi}{H}\left(-\frac{\partial E}{\partial v}+F\begin{Bmatrix}12\\2\end{Bmatrix}-E\begin{Bmatrix}22\\2\end{Bmatrix}\right)\right]$$
$$+\frac{X\Phi}{H}(FD'-ED'')-\frac{M}{\mathcal{H}}\frac{\partial X}{\partial v}. \tag{16}$$

where the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}$ are formed with respect to the square of the linear element of S.

Again, from the first of formulæ (4) we get

$$\frac{\partial^2 x_1}{\partial u\,\partial v}=\left[\frac{\partial}{\partial v}\left(\frac{\phi D}{HK}\right)+\frac{\phi D}{HK}\begin{Bmatrix}22\\2\end{Bmatrix}'-\frac{\phi D'}{HK}\begin{Bmatrix}11\\2\end{Bmatrix}'-\frac{M}{HK}\right]\frac{\partial X}{\partial v}$$
$$+\left[\frac{\phi D}{HK}\begin{Bmatrix}22\\1\end{Bmatrix}'-\frac{\partial}{\partial v}\left(\frac{D'\phi}{HK}\right)-\frac{\phi D'}{HK}\begin{Bmatrix}11\\1\end{Bmatrix}'\right]\frac{\partial X}{\partial u}$$
$$-\left[\frac{\partial}{\partial v}\left(\frac{M}{HK}\right)+\frac{\Phi}{HK}(D\mathcal{G}+\mathcal{E}D')\right]X, \tag{17}$$

from which it follows that

$$\sum X\frac{\partial^2 x_1}{\partial u\,\partial v}=-\left[\frac{\partial}{\partial v}\left(\frac{M}{HK}\right)+\frac{\Phi}{H}(-D'F+ED'')\right]. \tag{18}$$

Making use of the second of (4), we find

$$\sum X\frac{\partial^2 x_1}{\partial u\,\partial v}=-\left[\frac{\partial}{\partial u}\left(\frac{N}{HK}\right)+\frac{\Phi}{H}\left(-GD+FD'\right)\right], \tag{19}$$

and in a similar way

$$\left.\begin{aligned}\sum X\frac{\partial^2 x_1}{\partial u^2}&=-\left[\frac{\partial}{\partial u}\left(\frac{M}{HK}\right)+\frac{\Phi}{H}(ED'-FD)\right],\\ \sum X\frac{\partial^2 x_1}{\partial v^2}&=-\left[\frac{\partial}{\partial v}\left(\frac{N}{HK}\right)+\frac{\Phi}{H}(FD''-GD')\right].\end{aligned}\right\} \tag{20}$$

We are in a position now to calculate the second fundamental coefficients D_1, D_1', D_1'' of S_1. By definition,

$$D_1'=\sum X_1\frac{\partial^2 x_1}{\partial u\,\partial v}=\sum\frac{A_1}{H_1}\frac{\partial^2 x'}{\partial u\,\partial v}.$$

Substituting the expressions for X_1 and $\dfrac{\partial^2 x'}{\partial u\,\partial v}$ as given by (15) and (16), and

making use of (18), we can reduce this to the form:

$$D_1' = \frac{\phi H}{\delta}\left[\phi K(FD' - ED'') - HK\frac{\partial}{\partial v}\frac{M}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\right].$$

By differentiating the second of formulæ (6) with respect to u, another expression for $\frac{\partial^2 x_1}{\partial u\,\partial v}$ can be obtained. Proceeding with this as in the previous case, and making use of (19), we find

$$D_1' = \frac{\phi H}{\delta}\left[\phi K(GD - FD') - HK\frac{\partial}{\partial u}\frac{N}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\right].$$

From these two expressions for D_1', those for D_1 and D_1'' can be written by analogy. Replacing δ by its equivalent (14), we have the following expressions for the second fundamental coefficients of S_1:

$$\left.\begin{aligned}
D_1 &= \frac{\phi^2}{H_1 K}\left[\phi K(FD - ED') - HK\frac{\partial}{\partial u}\frac{M}{HK} + M\begin{Bmatrix}11\\1\end{Bmatrix} + N\begin{Bmatrix}11\\2\end{Bmatrix}\right],\\
D_1' &= \frac{\phi^2}{H_1 K}\left[\phi K(FD' - ED'') - HK\frac{\partial}{\partial v}\frac{M}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\right],\\
&= \frac{\phi^2}{H_1 K}\left[\phi K(GD - FD') - HK\frac{\partial}{\partial u}\frac{N}{HK} + M\begin{Bmatrix}12\\1\end{Bmatrix} + N\begin{Bmatrix}12\\2\end{Bmatrix}\right],\\
D_1'' &= \frac{\phi^2}{H_1 K}\left[\phi K(GD' - FD'') - HK\frac{\partial}{\partial v}\frac{N}{HK} + M\begin{Bmatrix}22\\1\end{Bmatrix} + N\begin{Bmatrix}22\\2\end{Bmatrix}\right].
\end{aligned}\right\} \quad (21)$$

By introducing the values for the partial differential quotients of D, D', D'', as given by the Weingarten formulæ, these expressions can be brought to the form

$$\left.\begin{aligned}
D_1 &= \frac{\phi^2}{H_1 K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{E}\phi\Big)\\
&\qquad\qquad - D\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{F}\phi\Big)\Big],\\
D_1' &= \frac{\phi^2}{H_1 K}\Big[D'\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{F}\phi\Big)\\
&\qquad\qquad - D\Big(\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{G}\phi\Big)\Big],\\
&= \frac{\phi^2}{H_1 K}\Big[D''\Big(\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{E}\phi\Big)\\
&\qquad\qquad - D'\Big(\frac{\partial^2\phi}{\partial u\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{F}\phi\Big)\Big],\\
D_1'' &= \frac{\phi^2}{H_1 K}\Big[D''\Big(\frac{\partial^2\phi}{\partial u\,\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{F}\phi\Big)\\
&\qquad\qquad - D'\Big(\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{G}\phi\Big)\Big].
\end{aligned}\right\} \quad (22)$$

These expressions can be put in a simpler and more convenient form by the introduction of the fundamental coefficients of a third surface S_0, which is the *associate* surface of S in an infinitesimal deformation of the latter. As defined by Bianchi,* associate surfaces are such that the normals to the surfaces at corresponding points are parallel and the characteristic function of an infinitesimal deformation of one is equal to the distance from the origin of coordinates to the tangent plane to the other. Denote by D_0, D_0', D_0'' the second fundamental coefficients of S_0; then*

$$
\begin{aligned}
D_0 &= -\left(\frac{\partial^2\phi}{\partial u^2} - \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{E}\phi\right),\\
D_0' &= -\left(\frac{\partial^2\phi}{\partial u\,\partial v} - \begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{F}\phi\right),\\
D_0'' &= -\left(\frac{\partial^2\phi}{\partial v^2} - \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u} - \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v} + \mathcal{G}\phi\right).
\end{aligned}
\tag{23}
$$

By means of these expressions, the formulæ (22) can be written:

$$
\begin{aligned}
D_1 &= \frac{\phi^2}{H_1K}(DD_0' - D_0D'),\\
D_1' &= \frac{\phi^2}{H_1K}(DD_0'' - D_0'D') = \frac{\phi^2}{H_1K}(D'D_0' - D''D_0),\\
D_1'' &= \frac{\phi^2}{H_1K}(D'D_0'' - D_0'D'').
\end{aligned}
\tag{24}
$$

From these values it can be shown at once that

$$DD_1'' + D_1D'' - 2D'D_1' = 0. \tag{25}$$

Hence,

When two surfaces correspond by orthogonality of linear elements, asymptotic lines on one correspond to a conjugate system on the other.†

When two such surfaces are referred to this system of lines, the equations (24) reduce to a form from which it is readily seen that when S is a surface of positive curvature, S_1 is a surface of negative curvature; and when S has negative curvature at a point, the curvature of S_1 at the corresponding point can be positive or negative.

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* Lezioni, p. 278.

† Bianchi, p. 284.

Since the expressions for D_1, D_1', D_1'' are linear in the second fundamental coefficients of either S or S_0, the latter can be readily found by solving equations (24). Thus:

$$\left.\begin{aligned} D_0 &= \frac{H_1 H^2}{\phi^2}(D_1 D' - D_1' D), \\ D_0' &= \frac{H_1 H^2}{\phi^2}(D_1 D'' - D_1' D') = \frac{H_1 H^2}{\phi^2}(D_1' D' - D'' D), \\ D_0'' &= \frac{H_1 H^2}{\phi^2}(D_1' D'' - D_1'' D'), \end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned} D &= \frac{H_1 H_0^2 K}{\phi^2 K_0}(D_1' D_0 - D_1 D_0'), \\ D' &= \frac{H_1 H_0^2 K}{\phi^2 K_0}(D_1' D_0' - D_1 D_0'') = \frac{H_1 H_0^2 K}{\phi^2 K_0}(D_1'' D_0 - D_1' D_0'), \\ D'' &= \frac{H_1 H_0^2 K}{\phi^2 K_0}(D_1'' D_0' - D_1' D_0''), \end{aligned}\right\} \tag{27}$$

where H_0 and K are the functions for S_0 corresponding to H and K for S.

From (24) and (26) we get at once

$$\left.\begin{aligned} D_0 D_1'' + D_0'' D_1 - 2D_1' D_0' = 0, \\ D_0 D'' + D_0'' D - 2D_0' D' = 0. \end{aligned}\right\} \tag{28}$$

Recalling the interpretation of equation (25), we have the theorem:

The asymptotic lines on any one of the three surfaces, S, S_1, S_0, correspond to the system of lines on the other two surfaces which is conjugate for both.

When S is referred to the conjugate system corresponding to the asymptotic lines on S_0, the following relations result, since tangent planes to the two surfaces are parallel:

$$\frac{\partial x}{\partial u} = \frac{D}{D_0'}\frac{\partial x_0}{\partial v}, \quad \frac{\partial x}{\partial v} = \frac{D''}{D_0'}\frac{\partial x_0}{\partial u}, \tag{29}$$

and analogous expressions in y and z. From these follow

$$E = \frac{D^2}{D_0'^2} G_0, \quad F = \frac{DD''}{D_0'^2} F_0, \quad G = \frac{D''^2}{D_0'^2} E_0. \tag{30}$$

From (29) it is manifest that the asymptotic directions on S_0 are parallel to those of the corresponding conjugate systems on S. Again, since S and S_0 correspond by parallelism of tangent planes, the directions of the corresponding conjugate systems on the two surfaces are parallel.

In the special case where the lines of curvature on S correspond to a conjugate system on S_1, it is readily seen from above remarks that the asymptotic lines on S_0 form an orthogonal system. Since the converse is also true, we have the result due to Weingarten:*

The necessary and sufficient condition that the lines of curvature on S correspond to a conjugate system on S_1 is that S_0 be a minimal surface, or, what is the same thing, the spherical representation of the former be isothermal.

An expression for the total curvature of S_1 can be obtained at once by means of (24). It is

$$K_1 = \frac{H^2\phi^4 H_0^2}{H_1^4}\,\frac{K_0}{K}. \tag{31}$$

From this expression and a previous paragraph we get the theorem:

When S is a surface of positive curvature, S_1 and S_0 are of negative curvature; when S is of negative curvature, S_1 and S_0 have opposite kinds of curvature at corresponding points.†

The general expression for the mean curvature of S_1 can be found from formulæ (7) and (24); however, we will consider only the special case where the parametric lines on S_1 correspond to asymptotic lines on S_0. Recalling that in general

$$L_1 = \frac{E_1 D_1'' + G_1 D_1 - 2F_1 D_1'}{H_1^2}, \tag{32}$$

where L_1 denotes the mean curvature of S_1, we find in the case mentioned,

$$L_1 = \frac{\phi^2 D_0'}{H_1^3 K}\left\{\phi^2(GD - ED'') + \frac{D_0'^2}{\mathcal{H}^2}\left[\frac{DG}{E_0}\left(\frac{\partial\phi}{\partial u}\right)^2 - \frac{ED''}{G_0}\left(\frac{\partial\phi}{\partial v}\right)^2\right]\right\}. \tag{33}$$

The fundamental coefficients of the spherical representation of S_1 can be found directly from formulæ (7) and (24); they are

$$\begin{aligned}
\mathfrak{E}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 E_0 + \frac{1}{H_0^2 K_0^2}\left(D_0\frac{\partial\phi}{\partial v} - D_0'\frac{\partial\phi}{\partial u}\right)^2\right],\\
\mathfrak{F}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 F_0 + \frac{1}{H_0^2 K_0^2}\left(D_0\frac{\partial\phi}{\partial v} - D_0'\frac{\partial\phi}{\partial u}\right)\left(D_0'\frac{\partial\phi}{\partial v} - D_0''\frac{\partial\phi}{\partial u}\right)\right],\\
\mathfrak{G}_1 &= \frac{\phi^4 H^4}{H_1^4}\left[\phi^2 G_0 + \frac{1}{H_0^2 K_0^2}\left(D_0'\frac{\partial\phi}{\partial v} - D_0''\frac{\partial\phi}{\partial u}\right)^2\right].
\end{aligned} \tag{34}$$

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* Sitzungsberichte der König. Akademie zu Berlin, 1886; Bianchi, p. 285.

† The ideas of this theorem are set forth by Bianchi, pp. 279, 286.

We shall denote by $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_1$ the Christoffel symbols formed with respect to the square of the linear element of S_1. Consider first $\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1$. Substituting the values of E_1, F_1, G_1 from (7) and making some straightforward reductions, we get

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1 = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{ND'}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} \left[D'(MG + NF) D'_0 - NFD''D_0 - MGDD''_0 \right].$$

The value for $\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1$ can be written down at once by analogy. By proceeding in ways similar to the above, the expressions for $\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1$ and $\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1$ can be found, and then $\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1$ and $\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1$ follow by analogy. Having effected these operations, we get the following:

$$\left. \begin{aligned} \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} - \frac{ND}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} (DD'_0 - D_0 D')(NF - GM), \\ \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} - \frac{MD}{(DD'' - D'^2)\phi} + \frac{\phi^2}{\mathcal{K}^2 H_1^2} (D'D_0 - DD'_0)(EN - FM), \end{aligned} \right\} \quad (35)$$

$$\left. \begin{aligned} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \frac{ND'}{(DD'' - D'^2)\phi} \\ &\quad + \frac{\phi^2}{\mathcal{K}^2 H_1^2} \left[(MG + NF) D'_0 D' - NFD''D_0 - MGDD''_0 \right], \\ \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} + \frac{MD'}{(DD'' - D'^2)\phi} \\ &\quad - \frac{\phi^2}{\mathcal{K}^2 H_1^2} \left[(NE + MF) D'_0 D' - NED''D_0 - MFDD''_0 \right], \end{aligned} \right\} \quad (36)$$

$$\left. \begin{aligned} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} + \frac{ND''}{(DD'' - D'^2)\phi} - \frac{\phi^2}{\mathcal{K}^2 H_1^2} \left[(FN - GM)(D''D'_0 - D'D''_0) \right], \\ \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1 &= \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} + \frac{MD''}{(DD'' - D'^2)\phi} - \frac{\phi^2}{\mathcal{K}^2 H_1^2} \left[(FM - EN)(D''D'_0 - D'D''_0) \right]. \end{aligned} \right\} \quad (37)$$

Consider the case where S_1 is referred to the conjugate system corresponding to the asymptotic lines on S; then

$$D = D'' = D'_0 = D'_1 = 0.$$

From (36) we have, on replacing M and N by their values,

$$\begin{Bmatrix}12\\1\end{Bmatrix}_1=\begin{Bmatrix}12\\1\end{Bmatrix}+\frac{\partial\log\phi}{\partial v},\quad \begin{Bmatrix}12\\2\end{Bmatrix}_1=\begin{Bmatrix}12\\2\end{Bmatrix}+\frac{\partial\log\phi}{\partial u}.$$

Since S is referred to its asymptotic lines, we must have

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}=\frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}.$$

Hence, if the above equations are differentiated with respect to u and v respectively, and this condition is taken into account, it is necessary that

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}_1=\frac{\partial}{\partial v}\begin{Bmatrix}12\\1\end{Bmatrix}_1.$$

But when this is true, the equation of the point coordinates of S_1, referred to this conjugate system, namely,

$$\frac{\partial^2\theta}{\partial u\,\partial v}-\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\theta}{\partial u}-\begin{Bmatrix}12\\2\end{Bmatrix}_1\frac{\partial\theta}{\partial v}=0,$$

has equal invariants. Since this result is entirely reciprocal as regards S and S_1, we have the following theorem:*

If two surfaces S and S_1 correspond by orthogonality of linear elements and one of the surfaces is referred to its asymptotic lines, the point equation relative to the corresponding conjugate system on the other has equal invariants.

Suppose S and S_1 are referred to the system of lines which is conjugate for both; then

$$D'=D_1'=D_0=D_0''=0.$$

From (36) we have

$$\begin{Bmatrix}12\\1\end{Bmatrix}_1=\begin{Bmatrix}12\\1\end{Bmatrix},\quad \begin{Bmatrix}12\\2\end{Bmatrix}_1=\begin{Bmatrix}12\\2\end{Bmatrix},$$

that is, the point coordinates of S_1 satisfy the same equation as those of S, namely,

$$\frac{\partial^2\theta}{\partial u\,\partial v}-\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\theta}{\partial u}-\begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial\theta}{\partial v}=0.$$

Again, we have that the tangential equation of the surface S, referred to this conjugate system, is

$$\frac{\partial^2\phi}{\partial u\,\partial v}-\begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\phi}{\partial u}-\begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\phi}{\partial v}+\mathcal{J}\phi=0,$$

* Darboux, Leçons, t. IV, p. 50; Bianchi, p. 286.

but since the spherical representation of S is also that of S_0 and the latter is referred to asymptotic lines, it follows that the tangential equation has equal invariants. Inasmuch as the relation between S and S_1 is reciprocal, this result is also true for S_1. Hence:

*If two surfaces correspond by orthogonality of linear elements they have the same point equation when the parametric lines on each constitute the conjugate system corresponding on the two surfaces; and the tangential equations of both surfaces relatively to this system have equal invariants.**

Darboux† has shown that when the condition

$$\sum dx\, dx_1 = 0$$

is satisfied by the cartesian coordinates of corresponding points on S and S_1 there exist three quantities x_0, y_0, z_0 which are such that

$$\begin{aligned} dx_1 &= z_0 dy - y_0 dz, \\ dy_1 &= x_0 dz - z_0 dx, \\ dz_1 &= y_0 dx - x_0 dy, \end{aligned} \tag{38}$$

and Genty‡ has proved that the surface which has x_0, y_0, z_0 for its point coordinates is S_0, the associate surface of S in an infinitesimal deformation of the latter.

Multiplying these equations by x_0, y_0, z_0 respectively and adding, we have

$$x_0 dx_1 + y_0 dy_1 + z_0 dz_1 = 0,$$

from which it follows that

$$\sum x_0 \frac{\partial x_1}{\partial u} = 0, \quad \sum x_0 \frac{\partial x_1}{\partial v} = 0.$$

But

$$\sum X_1 \frac{\partial x_1}{\partial u} = 0, \quad \sum X_1 \frac{\partial x}{\partial v} = 0,$$

hence,

$$\frac{x_0}{X_1} = \frac{y_0}{Y_1} = \frac{z_0}{Z_1} = r_0,$$

where r_0 is the radius vector of S_0 at the point (x_0, y_0, z_0). From this it follows

* The second part of this theorem is given by Darboux, IV, p. 71.

† Leçons, t. IV, p. 8.

‡ Toulouse Annales, t. IX, E.

that *the radius vector of S_0 at the point M_0 is parallel to the normal to S_1 at the corresponding point.**

And as a consequence of this:

The necessary and sufficient condition that the tangent planes at corresponding points of S and S_1 be parallel is that S_0 be a sphere.

III.

Consider the surface S', for which the cartesian coordinates of a point are defined by

$$x' = x + \tau x_1,\ y' = y + \tau y_1,\ z' = z + \tau z_1, \tag{1}$$

where (x, y, z) and (x_1, y_1, z_1) are the coordinates of corresponding points on S and S_1 and τ is a constant which, for the present, may have any finite value.

If ds' denote an element of length on S', the square of this linear element may be written in the form

$$ds'^2 = E' du^2 + 2F' du dv + G' dv_1^2 \tag{2}$$

where, in consequence of (1),

$$E' = E + \tau^2 E_1,\ F' = F + \tau^2 F_1,\ G' = G + \tau^2 G_1, \tag{3}$$

or by (II, 7)

$$\left.\begin{aligned} E' &= E(1 + \tau^2\phi^2) + \frac{\tau^2 M^2}{H^2K^2}, \\ F' &= F(1 + \tau^2\phi^2) + \frac{\tau^2 MN}{H^2K^2}, \\ G' &= G(1 + \tau^2\phi^2) + \frac{\tau^2 N^2}{H^2K^2}. \end{aligned}\right\} \tag{3'}$$

From (3) and (3′) we obtain directly

$$H'^2 = (H^2 + \frac{\tau^2}{\phi^2} H_1^2)(1 + \tau^2\phi^2), \tag{4}$$

and

$$H'^2 = (1 + \tau^2\phi^2)\left[H^2(1 + \tau^2\phi^2) + \frac{\tau^2(EN^2 - 2FMN + GM^2)}{H^2K^2}\right]. \tag{5}$$

Put

$$A' = \frac{\partial(y', z')}{\partial(u, v)},\quad B' = \frac{\partial(z', x')}{\partial(u, v)},\quad C' = \frac{\partial(x', y')}{\partial(u, v)}; \tag{6}$$

* This theorem is essentially the same as that given by Darboux, IV, p. 59.

then if the differential quotients of x', y', z' be replaced by their expressions from (1) and formulæ (II, 6) be used, A' can be expressed in the form

$$A' = A + \frac{\tau}{K}\,\frac{\partial(\phi,\,X)}{\partial(u,\,v)} + \tau^2 A_1,$$

and the expressions for B' and C' are similar.

If the second fundamental coefficients of S' be denoted by $D^{(1)}$, $D'^{(1)}$, $D''^{(1)}$, the first is given by

$$D^{(1)} = \sum X'\frac{\partial^2 x'}{\partial u^2} = \sum \frac{A'}{H'}\Big(\frac{\partial^2 x}{\partial u^2} + \tau\frac{\partial^2 x_1}{\partial u^2}\Big).$$

Making use of formulæ (II, 13 and 20) we can bring this to the form

$$D^{(1)} = \frac{1}{H'}\Big[\frac{D}{H}\Big(H^2 + \frac{H_1^2}{\phi^2}\tau^2\Big) + H_1 D_1 \tau\Big(\frac{1}{\phi^2} + \tau^2\Big)\Big]. \tag{7}$$

The expressions for $D'^{(1)}$ and $D''^{(1)}$ follow by analogy; they are

$$D'\ ) = \frac{1}{H'}\Big[\frac{D'}{H}\Big(H^2 + \frac{H_1^2}{\phi^2}\tau^2\Big) + H_1 D_1'\tau\Big(\frac{1}{\phi^2} + \tau^2\Big)\Big],$$
$$D''^{(1)} = \frac{1}{H^2}\Big[\frac{D''}{H}\Big(H^2 + \frac{H_1^2}{\phi^2}\tau^2\Big) + H_1 D_1''\tau\Big(\frac{1}{\phi^2} + \tau^2\Big)\Big]. \tag{7}$$

From these expressions we can readily find an expression for the total curvature of S' by taking account of the condition (II, 25). Thus:

$$K' = \frac{K}{(1+\tau^2\phi^2)^2} + \frac{\tau^2 H_1^4 K_1\Big(\tau^2 + \frac{1}{\phi^2}\Big)^2}{(1+\tau^2\ {}^2)^2\Big(H^2 + H_1^2\frac{\tau^2}{\phi^2}\Big)}. \tag{8}$$

In the special case where S_1 is a sphere, the distance between corresponding points on the two applicable surfaces Σ_1 and Σ_2, defined by

$$\xi_1 = x + \tau x_1, \quad \eta_1 = y + \tau y_1, \quad \zeta_1 = z + \tau z_1,$$
$$\xi_2 = x - \tau x_1, \quad \eta_2 = y - \tau y_1, \quad \zeta_2 = z - \tau z_1,$$

where $\tau = \sqrt{x_1^2 + y_1^2 + z_1^2}$, is constant.

It is evident that in this case the surface Σ_1 is the same as S' in the beginning of this section, and that the coefficients of Σ_2 are gotten by changing τ into

$-\tau$ in those formulæ. From the expressions for E', F', G', it is seen that the only case in which Σ_1 and Σ_2 could be applicable to S would arise from the value

$$\phi = 0,$$

which simply means that Σ_1, Σ_2 and S coincide.

In order that Σ_1 and Σ_2 be the same surface to a displacement *près*, it is necessary and sufficient that their second fundamental coefficients are equal. Referring to formulæ (7), we remark that we must have

$$\frac{1}{\phi^2} + \tau^2 = 0,$$

that is, $\phi = \text{const.}$, from which it follows that S_0 may be either a sphere or a plane. However, the former case must be excluded, since S_0 and S_1 cannot be spheres simultaneously. But when S_0 is a plane, S also is a plane; hence,

The necessary and sufficient condition that the two surfaces Σ_1 and Σ_2 be the same surface to a displacement près, is that S be a plane.

The formulæ which have been deduced in this section are evidently independent of the nature of the constant quantity τ. If, then, τ is replaced by ε, which is a small constant whose powers higher than the first are negligible, the surface S' will be infinitely near S and will be obtained by deforming points of the latter along lines whose direction cosines are proportional to the cartesian coordinates of the corresponding points on S_1. With Darboux we shall call the lines, in the direction of which the deformation is effected, the *directrices of the deformation.*

The square of the linear element of S' is

$$ds'^2 = dx^2 + dy^2 + dz^2 + \varepsilon^2 (dx_1^2 + dy_1^2 + dz_1^2),$$

that is, S' is applicable to S to terms of the second order *près*, which, by hypothesis, are neglected. Hence,

$$E' = E, \quad F' = F, \quad G' = G. \tag{9}$$

In this case the expressions for the second fundamental coefficients of S' become

$$D^{(1)} = D + \varepsilon \frac{H_1 D_1}{\phi^2 H}, \quad D'^{(1)} = D' + \varepsilon \frac{H_1 D_1'}{\phi^2 H}, \quad D''^{(1)} = D'' + \varepsilon \frac{H_1 D_1''}{\phi^2 H}, \tag{10}$$

and we have for the total and mean curvature of S' respectively,

$$K' = K, \quad L' = L + \varepsilon \frac{H_1(GD_1 + ED_1'' - 2FD_1')}{\phi^2 H^3},$$

the former being simply an expression of the Gauss theorem.

The expressions for the fundamental coefficients of the spherical representation of S', denoted by $\mathfrak{E}'$, $\mathfrak{F}'$, $\mathfrak{G}'$, are readily calculated; they are

$$\mathfrak{E}' = \mathfrak{E} + \frac{2H_1\varepsilon}{\phi^2 H^3}[GDD_1 - F(DD_1' + D'D_1) + ED'D_1'],$$

$$\mathfrak{F}' = \mathfrak{F} + \frac{2H_1\varepsilon}{\phi^2 H^3}[G(DD_1' + D_1D') - F(DD_1'' + D''D_1 + 2D'D_1') + E(D'D_1'' + D''D_1')],$$

$$\mathfrak{G}' = \mathfrak{G} + \frac{2H_1\varepsilon}{\phi^2 H^3}[GD'D_1' - F(D'D_1'' + D_1'D'') + ED_1''D''].$$

Replace D_1, D_1', D_1'' by their values as given in (24); then

$$\left.\begin{aligned} \mathfrak{E}' &= \mathfrak{E} + \frac{2\varepsilon}{HK}\left(\mathfrak{E}D_0' - \mathfrak{F}D_0\right), \\ \mathfrak{F}' &= \mathfrak{F} + \frac{2\varepsilon}{HK}\left(\mathfrak{E}D_0'' - \mathfrak{G}D_0\right), \\ \mathfrak{G}' &= \mathfrak{G} + \frac{2\varepsilon}{HK}\left(\mathfrak{F}D_0'' - \mathfrak{G}D_0'\right). \end{aligned}\right\} \tag{11}$$

It is evident from these expressions that the necessary and sufficient condition that the tangent planes at corresponding points of S and S' be parallel is expressed by

$$\frac{\mathfrak{E}}{D_0} = \frac{\mathfrak{F}}{D_0'} = \frac{\mathfrak{G}}{D_0''},$$

or, since the spherical representation of S and S_0 is the same,

$$\frac{\mathfrak{E}_0}{D_0} = \frac{\mathfrak{F}_0}{D_0'} = \frac{\mathfrak{G}_0}{D_0''}.$$

But this condition is satisfied only when S_0 is a plane or a sphere. In the former case, S and S' are planes, that is, S has been subjected to a displacement and not an infinitesimal deformation; hence, this case is excluded. We have then the following theorem:

The necessary and sufficient condition that S and S' correspond by parallelism of tangent planes is that S_0 be a sphere, or what is the same thing, that S and S_1 have such correspondence.

From the expression for $D'^{(1)}$, it is manifest that the conjugate system on S, which corresponds to a conjugate system on S', is that which corresponds to a conjugate system on S_1, or, what is the same thing, to the asymptotic lines on S_0. Hence :*

The conjugate system of lines on S which remains conjugate for a given infinitesimal deformation of the surface corresponds to the asymptotic lines on its associate.

If $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}^{(1)}$ denotes the Christoffel symbols formed with respect to the square of the linear element of S', we have in consequence of (9),

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}^{(1)} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}, \quad \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}^{(1)} = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}.$$

Hence, *when S, S_1 and S' are referred to the double system* of *lines which are conjugate for all three surfaces, their point coordinates are particular solutions of the same partial differential equation of the second order.*

When the lines of curvature constitute the system which has for its correspondent a conjugate system on S', the latter is also made up of lines of curvature. Recalling a result obtained in a previous section, we have the well-known theorem :†

The necessary and sufficient condition that the lines of curvature of a surface be preserved in an infinitesimal deformation is that the spherical representation of the lines of curvature form an isothermal system.

Guichard has shown‡ that any surface can be brought into correspondence with a plane by orthogonality of linear elements. Hence an infinitesimal deformation of the plane can be effected as soon as any surface whatever has been brought into such correspondence by the means suggested by Guichard. It is evident that S_0 also will be a plane. From formulæ (10), it follows that on the resulting surface the asymptotic lines will correspond to the asymptotic lines on S_1.

If, now, the deformation of any surface is considered for which S_1 is a plane, the formulæ (10) show that the second fundamental coefficients of the original surface and the surface resulting from the deformation are equal, since $\phi \neq 0$ for this case. Hence, when S_1 is a plane, there is no deformation, but only an infinitesimal displacement of S. We shall show that this sufficient condition is also necessary.

* Bianchi, p. 284. † Ib., p. 285. ‡ Comptes Rendus, t. CXIV, p. 729.

In order that S' differs from S only by an infinitely small displacement, it is necessary that

$$x_1 = c_1 + \gamma y - \beta z, \quad y_1 = c_2 + \alpha z - \gamma x, \quad z_1 = c_3 + \beta x - \alpha y, \tag{12}$$

where $\alpha, \beta, \gamma, c_1, c_2, c_3$ are constants. Eliminating x, y, z from these expressions, we find that S_1 is a plane. Hence,*

The necessary and sufficient condition that S' be S to an infinitely small displacement is that S_1 be a plane.

Differentiating the expressions (12) and comparing them with the formulæ (II, 38), we find

$$x_0 = \alpha, \quad y_0 = \beta, \quad z_0 = \gamma,$$

that is, *when S_1 is a plane, S_0 is a point, and conversely.*

Let S be referred to its asymptotic lines. Since then $D_1' = 0$, in order that the corresponding lines on S' be the asymptotic lines, either

$$D_1 = D_1' = D_1'' = 0,$$

or

$$\varepsilon = 0.$$

It has been shown that in the former case there is only a displacement; in latter case, there has been no deformation. Hence,

The asymptotic lines of a surface are not preserved in any infinitesimal deformation of the surface.

IV.

From the form of the characteristic equation, it is readily seen that only in the case where S is a minimal surface can the characteristic function ϕ be a constant. When ϕ is a constant, S_0 is either a plane or a sphere. In the former case, S is a plane, and it will be remarked that this case is in perfect accord with the results which follow when S_0 is a sphere.

Let S_0 be a sphere of radius unity, that is, $\phi = 1$. Formulæ (II, 7) give

$$E_1 = E, \quad F_1 = F, \quad G_1 = G,$$

hence,

$$ds_1^2 = ds^2,$$

* Darboux, IV, p. 9.

that is, S and S_1 are applicable surfaces. Since S and S_1 correspond by orthogonality of linear elements, S_1 is the *adjoint minimal* surface of S. This is immediate from the following theorem, as stated by Darboux:*

If two surfaces are applicable to one another by orthogonality of corresponding linear elements, they can only be two minimal surfaces, of which one is the adjoint of the other.

This result is also obtained by remarking that in this case x_0, y_0, z_0 may be replaced by X, Y, Z respectively, so that formulæ (II, 38) become

$$\begin{aligned} dx_1 &= Ydz - Zdy, \\ dy_1 &= Zdx - Xdz, \\ dz_1 &= Xdy - Ydx, \end{aligned}$$

which are the formulæ of Schwarz,† giving the relations between the differentials of the cartesian coordinates of a minimal surface and of its adjoint and the direction cosines of tangent planes.

Since the adjoint of a minimal surface is unique, we have the following theorem as a consequence of the above theorem of Darboux:‡

The necessary and sufficient condition that a surface S be applicable to a surface S_1 which figures in an infinitesimal deformation of the former, is that S_1 correspond to the value unity of the characteristic function; moreover, this is only possible when S is a minimal surface.

From the formulæ (II, 21), we obtain the following relations between the fundamental coefficients of a minimal surface and its adjoint when referred to a general system of curvilinear coordinates.

$$\left.\begin{aligned} D_1 &= \frac{FD - ED'}{H}, \quad D_1'' = \frac{GD' - FD''}{H}, \\ D_1' &= \frac{FD' - ED''}{H} = \frac{GD - FD'}{H}. \end{aligned}\right\} \tag{1}$$

Since S_0 is a sphere,

$$D_0,\ D_0',\ D_0'' = -E_0,\ -F_0,\ -G_0;$$

and remarking that in the present case S_0 is the spherical representation of S, we have from (II, 24) the following relations between the second fundamental

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* Leçons, t. I, p. 331.

† "Miscellen aus dem Gebiete der Minimalflachen," Crelle, 80, p. 287.

‡ Bianchi, p. 348.

coefficients of a minimal surface and its adjoint and the coefficients of their common spherical representation:

$$\left.\begin{aligned} D_1 &= \frac{D'\mathcal{E}-D\mathcal{F}}{\mathcal{H}}, \quad D_1'' = \frac{\mathcal{F}D''-\mathcal{G}D'}{\mathcal{H}}, \\ D_1' &= \frac{\mathcal{E}D''-\mathcal{F}D'}{\mathcal{H}} = \frac{\mathcal{F}D'-\mathcal{G}D}{\mathcal{H}}. \end{aligned}\right\} \tag{2}$$

Comparing the two sets of expressions (1) and (2), we find that in the case of a minimal surface,

$$\frac{\mathcal{E}}{E} = \frac{\mathcal{F}}{F} = \frac{\mathcal{G}}{G} = -K; \tag{3}$$

from this it follows that

$$d\sigma^2 = -Kds^2 = -Kds_1^2. \tag{4}$$

In the present case the asymptotic directions on S_0 are given by

$$\mathcal{E}du^2 + 2\mathcal{F}du\,dv + \mathcal{G}dv^2 = 0,$$

that is, lines of length zero. Since the corresponding systems of lines on S and S_1 are conjugate for both surfaces, it follows from (4) that—

*The conjugate system of lines on a minimal surface which corresponds to a conjugate system on its adjoint, is made up of lines of length zero on both.**

When S_0, a sphere, is referred to its lines of length zero, the point coordinates are the following functions of the parameters of these lines:†

$$x_0 = \frac{u+v}{1+uv}, \quad y_0 = i\,\frac{u-v}{1+uv}, \quad z_0 = \frac{uv-1}{1+uv}. \tag{5}$$

However, since S_0 is the spherical representation of S, the direction cosines of the tangent plane to the latter are given by

$$X = \frac{u+v}{1+uv}, \quad Y = i\,\frac{u-v}{1+uv}, \quad Z = \frac{uv-1}{1+uv},$$

where u and v refer to conjugate system on S composed of lines of length zero.

In section III we showed that the necessary and sufficient condition that the tangent planes at corresponding points of S and S' be parallel is that S_0 be a sphere. Since the deformation of S into S' is such that S and S' are applicable,‡ we have that in the case where *S_1 is the adjoint of S, the surface S' is an associate minimal surface of S.* Conversely, when S and S' are associate minimal sur-

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* Darboux, Leçons, t. IV, p. 96. † Ib., I, p. 37. ‡ Darboux, Leçons, t. I, p. 326.

faces, S_0 is a sphere; but we cannot take a special value for its radius: hence it will be necessary to ascertain the relations between S and S_1. From (II, 21) it follows that S_1 also is a minimal surface, and since S_0 is a sphere, the tangent planes to S_1 and S_0 at corresponding points are parallel. Referring to formulæ (II, 7) we get

$$E_1 = E\phi^2, \quad F_1 = F\phi^2, \quad G_1 = G\phi^2.$$

Denote by subscript a the quantities belonging to the adjoint of S, then

$$X_1 = X = X_a, \quad Y_1 = Y = Y_a, \quad Z_1 = Z = Z_a,$$

and

$$E_1 = \phi^2 E = \phi^2 E_a, \quad F_1 = \phi^2 F = \phi^2 F_a, \quad G_1 = \phi^2 G = \phi^2 G_a\,;$$

and by making use of (II, 24) and (2), we find

$$D_1 = \phi D_a, \quad D_1' = \phi D_a', \quad D_1'' = \phi D_a''.$$

From the last and first of these three systems of equalities, we have, for $\phi = \text{const.}$,

$$\sum \frac{\partial X}{\partial u}\,\frac{\partial x_1}{\partial u} = \phi \sum \frac{\partial X}{\partial u}\,\frac{\partial x_a}{\partial u} = \sum \frac{\partial X}{\partial u}\,\frac{\partial \phi x_a}{\partial u},$$

whence

$$\sum \frac{\partial X}{\partial u}\,\frac{\partial (x_1 - \phi x_a)}{\partial u} = 0, \quad \sum \frac{\partial X}{\partial v}\,\frac{\partial (x_1 - \phi x_a)}{\partial u} = 0$$

Solving these two equations, we get

$$\frac{\dfrac{\partial (x_1 - \phi x_a)}{\partial u}}{\dfrac{\partial (Y, Z)}{\partial (u, v)}} = \frac{\dfrac{\partial (y_1 - \phi y_a)}{\partial u}}{\dfrac{\partial (Z, X)}{\partial (u, v)}} = \frac{\dfrac{\partial (z_1 - \phi z_a)}{\partial u}}{\dfrac{\partial (X, Y)}{\partial (u, v)}}\,;$$

and hence,

$$\frac{\dfrac{\partial (x_1 - \phi x_a)}{\partial u}}{X} = \frac{\dfrac{\partial (y_1 - \phi y_a)}{\partial u}}{Y} = \frac{\dfrac{\partial (z_1 - \phi z_a)}{\partial u}}{Z}.$$

Since

$$\sum X \frac{\partial x_1}{\partial u} = 0 \quad \text{and} \quad \sum X \frac{\partial x_a \phi}{\partial u} = 0,$$

we have

$$\sum X \frac{\partial (x_1 - x_a \phi)}{\partial u} = 0.$$

Combining this with the above, we get

$$\frac{\partial (x_1 - \phi x_a)}{\partial u} = \frac{\partial (y_1 - \phi y_a)}{\partial u} = \frac{\partial (z_1 - \phi z_a)}{\partial u} = 0,$$

and, in like manner, by considering $\sum \frac{\partial X}{\partial v} \frac{\partial x_1}{\partial v}$,

$$\frac{\partial (x_1 - \phi x_a)}{\partial v} = \frac{\partial (y_1 - \phi y_a)}{\partial v} = \frac{\partial (z_1 - \phi z_a)}{\partial v} = 0.$$

Hence, $$x_1 = \phi x_a + a, \quad y_1 = \phi y_a + b, \quad z_1 = \phi z_a + c,$$

where a, b, c are constants. From this it follows that—

If two surfaces have their tangent planes at corresponding points parallel and their second fundamental coefficients are in constant ratio, the one is a homothetic of the other to within a translation.

Since the infinitesimal deformation of a surface S, where the direction cosines of the directrices of deformation are proportional to the coordinates of a surface homothetic to S_1, is the same as when the direction cosines are proportional to the point coordinates of the latter, we have the following theorem:*

The necessary and sufficient condition that a minimal surface be deformed into a minimal surface associate to itself, is that the direction cosines of the directrices of the infinitesimal deformation be proportional to the cartesian coordinates of the corresponding points on the adjoint minimal surface of the given one.

From Darboux† we have, that if the point coordinates of a minimal surface, referred to its conjugate system of lines of length zero, are given by

$$\begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned}$$

then the associate surfaces of this surface are given by

$$\begin{aligned} x &= e^{i\alpha} A(u) + e^{-i\alpha} A_1(v), \\ y &= e^{i\alpha} B(u) + e^{-i\alpha} B_1(v), \\ z &= e^{i\alpha} C(u) + e^{-i\alpha} C_1(v), \end{aligned}$$

each surface corresponding to a value of α. Furthermore, for the values $\alpha = 0$ and $\alpha = \pi/2$, these give the coordinates of the given surface and its adjoint respectively. If, then, α is allowed to vary continuously, we have a succession of minimal surfaces, applicable to the given one and to each other, extending, as it were, from the given surface to its adjoint. Moreover, from the nature of this process, it follows that this series of surfaces is unique.

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* Bianchi, p. 346.

† Leçons, t. I, p. 322.

We will consider now this series of associate minimal surfaces from the standpoint of infinitesimal deformation. From what precedes, it is clear that the associate surface of this series, infinitely near the given surface and corresponding to a value of α very near zero, can be gotten by deforming S infinitely little along directrices whose direction cosines are proportional to the cartesian coordinates of the adjoint; and, furthermore, it has been shown that this is the only way to get this surface by deformation. Consider in turn this surface and the third one in the series. The latter is an associate of the former and infinitely near it, and hence can be gotten by effecting upon the second surface an infinitesimal deformation, whose directrices have for direction cosines quantities proportional to the cartesian coordinates of the corresponding point on the adjoint of the second surface. Proceeding in this way step by step, the above series of surfaces can be obtained. We have, then, the following result:

The series of minimal surfaces, associate to a given minimal surface, which is obtained by varying continuously a parameter which enters in the expressions of the cartesian coordinates of a point on any surface, can be gotten by deforming the given surface and its successive associates infinitely little in directrices whose cosines are proportional to the point coordinates of the adjoint minimal surface of the surface deformed.

By means of (III, 13), the following relations can be found between the fundamental coefficients of a minimal surface, its adjoint and the associate surface which is the result of a given infinitesimal deformation of this surface:

$$D^{(1)} = D + \frac{D_1}{K}\varepsilon, \quad D'^{(1)} = D' + \frac{D_1'}{K}\varepsilon, \quad D''^{(1)} = D'' + \frac{D_1''}{K}\varepsilon. \tag{6}$$

Consider the case where S is a sphere, and let it be referred to the conjugate system corresponding to asymptotic lines on S_0. Since every conjugate system of lines on a sphere is an orthogonal system, and since the asymptotic directions on S_0 and the corresponding conjugate directrices on S are parallel, it follows that the asymptotic directrices on S_0 must be orthogonal, that is, S_0 is a minimal surface. We have, then, that *the associate surface of a sphere is a minimal surface.**

Recalling that for the sphere

$$D,\ D',\ D'' = -E,\ -F,\ -G,$$

* Darboux, IV, p. 96.

and substituting these values in the expression for the mean curvature of the surface corresponding to the sphere by orthogonality of linear elements (II, 33), we have

$$L_1 = \frac{\phi^2 D_0'^3}{H_1^3 K} EG \left[\frac{1}{G_0} \left(\frac{\partial \phi}{\partial v} \right)^2 - \frac{1}{E_0} \left(\frac{\partial \phi}{\partial u} \right)^2 \right].$$

Since the asymptotic lines form an isothermal system on a minimal surface, we have

$$E_0 = G_0 ;$$

hence, the necessary and sufficient condition that S_1 be a minimal surface is

$$\frac{\partial \phi}{\partial v} = \pm \frac{\partial \phi}{\partial u},$$

It will be seen that the following discussion will be just as general if we consider only the case where the sign in the above equation is plus. The general integral of this equation is a function of $u + v$; write it $\phi(u + v)$. From the definition of associate surface this function must be of such a form that it is the distance from the origin to the tangent plane to the minimal surface S_0. Since S_0 is referred to its asymptotic lines

$$\mathcal{F}_0 = 0, \quad \mathfrak{E}_0 = \mathcal{G}_0 = \frac{1}{\rho_0},$$

where

$$\frac{1}{\rho_0^2} = - K_0 ;$$

hence ϕ must satisfy the equation *

$$\phi'' + \frac{1}{\rho_0} \phi = 0,$$

where ϕ'' denotes the second derivative of ϕ with respect to $u + v$. From the form of this equation it is evident that for such a function ϕ to exist there must be a minimal surface for which ρ_0 is a function of $u + v$. We proceed to the determination of such a surface. Replace F by zero, E and G by ρ_0 and K by $-\frac{1}{\rho_0^2}$ in the Gauss equation,† then it is found that ρ_0 must satisfy the equation

$$\frac{d^2 \log \theta}{d(u + v)^2} = \frac{1}{\theta}.$$

This equation is readily integrated and gives

$$\rho_0 = \frac{\alpha^2}{\beta^2} \left(\frac{\beta^2}{2} e^{\frac{u+v}{2\alpha}} + e^{-\frac{u+v}{2\alpha}} \right)^2$$

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*Bianchi, p. 137 (37).

†Ib., p. 67 (19).

where α and β are constants. Since the Gauss and Codazzi equations are satisfied in the preceding results, there is a minimal surface which when referred to its asymptotic lines has for the square of its linear element the expression

$$ds^2 = \frac{\alpha^2}{\beta^2}\left(\frac{\beta^2}{2} e^{\frac{u+v}{2\alpha}} + e^{-\frac{u+v}{2\alpha}}\right)^2 (du^2 + dv^2).$$

Recalling the preceding results we have that only when S_0 is this particular minimal surface and S is the sphere of radius unity is S_1 a minimal surface.

Consider again the case where S is a minimal surface. Since the spherical representation of its lines of curvature form an isothermal system, S can be deformed so as to preserve these lines, and, as we have seen, S_0 is a minimal surface in this case. It is natural to inquire whether there is any special relation between the two minimal surfaces S and S_0. Refer the former to its lines of curvature; then

$$ds^2 = \rho\,(du^2 + dv^2),$$

where

$$\rho = \rho_1 = -\rho_2,$$

hence,

$$E = G = \frac{E}{D} = -\frac{G}{D''},$$

from which it follows that

$$D = 1, \quad D'' = -1.$$

The surface S_0 will be referred to its asymptotic lines, hence

$$ds_0^2 = \rho_0\,(du^2 + dv^2);$$

from this we get

$$E_0 = G_0, \quad D_0' = -1,$$

Substituting these values of D, D_0', D'' in formulæ (II, 30), we get

$$E = G_0 = E_0, \quad G = E_0 = G_0.$$

Thus S and S_0 are applicable and we know that asymptotic lines on S_0 correspond to lines of curvature on S. Hence:

The minimal surface which is the associate of a given minimal surface in the infinitesimal deformation of the latter, leaving the lines of curvature unaltered, is the adjoint of the given one.

We shall now discuss the question as to whether it is possible to deform S and S_1 in such a way that the surfaces resulting from the deformation shall be

in correspondence by orthogonality of linear elements. Denote by (x', y', z') and (x_1, y_1', z_1') corresponding points on these new surfaces; then we can write

$$x' = x + \varepsilon x_1, \quad x_1' = x_1 + \varepsilon_1 x_2, \tag{7}$$

and similar expressions for the y's and z's. For these two surfaces to have the desired correspondence, we must have

$$\sum dx' \, dx_1' = \sum dx \, dx_1 + \varepsilon \sum dx_1^2 + \varepsilon_1 \sum dx \, dx_2 + \varepsilon\varepsilon_1 \sum dx_1 \, dx_2 = 0.$$

Since

$$\sum dx \, dx_1 = 0 \quad \text{and} \quad \sum dx_1 \, dx_2 = 0,$$

it is necessary that

$$\varepsilon \sum dx_1^2 + \varepsilon_1 \sum dx \, dx_2 = 0;$$

that is, the following relations also must hold:

$$\varkappa \sum \left(\frac{\partial x_1}{\partial u}\right)^2 = \sum \frac{\partial x}{\partial u} \frac{\partial x_2}{\partial u}, \quad \varkappa \sum \left(\frac{\partial x_1}{\partial v}\right)^2 = \sum \frac{\partial x}{\partial v} \frac{\partial x_2}{\partial v},$$

$$2\varkappa \sum \frac{\partial x_1}{\partial u} \frac{\partial x_1}{\partial v} = \sum \frac{\partial x}{\partial u} \frac{\partial x_2}{\partial v} + \sum \frac{\partial x}{\partial v} \frac{\partial x_2}{\partial u}, \tag{8}$$

where $\varkappa = -\dfrac{\varepsilon}{\varepsilon_1}$, a constant.

From a proposition of Cauchy, that the integration by quadratures of a system of linear equations with second members, follows as soon as a general solution of the same system with second members zero has been obtained, it follows that when S_1 has been found, that is, when the system of equations

$$\sum \frac{\partial x}{\partial u} \frac{\partial x_1}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v} \frac{\partial x_1}{\partial v} = 0, \quad \sum \frac{\partial x}{\partial v} \frac{\partial x_1}{\partial u} + \sum \frac{\partial x}{\partial u} \frac{\partial x_1}{\partial v} = 0$$

has been integrated, we can integrate by quadratures the systems (8), in which x_1, y_1, z_1 have been given the values obtained by the previous integration. If the integrals so found satisfy, in conjunction with those of the previous system, the condition

$$\sum dx_1 \, dx_2 = 0,$$

we know that when the deformation indicated by (7) is effected, the resulting surfaces correspond by orthogonality of linear elements.

That there are such cases can be shown by looking at the question with the means discussed in the preceding pages. For, if we apply the formulæ of section II to the surface S_1, at one time in conjunction with S and the other with S_2, and we denote by ϕ_1 and ϕ_2 the characteristic functions in these two cases, we have (II, 6),

$$\frac{\partial x}{\partial u} = \frac{\phi_1}{H_1}\left(F_1\frac{\partial x_1}{\partial u} - E_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}M_1,$$
$$\frac{\partial x}{\partial v} = \frac{\phi_1}{H_1}\left(G_1\frac{\partial x_1}{\partial u} - F_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}N_1,$$

with similar expressions in y and z; and also,

$$\frac{\partial x_2}{\partial u} = \frac{\phi_2}{H_1}\left(F_1\frac{\partial x_1}{\partial u} - E_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}M_2,$$
$$\frac{\partial x_2}{\partial v} = \frac{\phi_2}{H_1}\left(G_1\frac{\partial x_1}{\partial u} - F_1\frac{\partial x_1}{\partial v}\right) - \frac{X_1}{H_1K_1}N_2.$$

Substituting these values in the above equations, we can reduce them to the following:

$$M_1M_2 + E_1H_1^2K_1^2(\phi_1\phi_2 - \kappa) = 0,$$
$$N_1N_2 + G_1H_1^2K_1^2(\phi_1\phi_2 - \kappa) = 0,$$
$$M_1N_2 + M_2N_1 + 2F_1H_1^2K_1^2(\phi_1\phi_2 - \kappa) = 0,$$

whence

$$\frac{M_1M_2}{E_1} = \frac{N_1N_2}{G_1} = \frac{M_1N_2 + M_2N_1}{2F_1}, \tag{9}$$

or, on writing the expressions for M_1, N_1, M_2, N_2,

$$\frac{\left(D_1\frac{\partial\phi_1}{\partial v} - D_1'\frac{\partial\phi_1}{\partial u}\right)\left(D_1\frac{\partial\phi_2}{\partial v} - D_1'\frac{\partial\phi_2}{\partial u}\right)}{E_1} = \frac{\left(D_1'\frac{\partial\phi_1}{\partial v} - D_1''\frac{\partial\phi_1}{\partial u}\right)\left(D_1'\frac{\partial\phi_2}{\partial v} - D_1''\frac{\partial\phi_2}{\partial u}\right)}{G_1},$$

$$= \frac{\left(D_1\frac{\partial\phi_1}{\partial v} - D_1'\frac{\partial\phi_1}{\partial u}\right)\left(D_1'\frac{\partial\phi_2}{\partial v} - D_1''\frac{\partial\phi_2}{\partial u}\right) + \left(D_1\frac{\partial\phi_2}{\partial v} - D_1'\frac{\partial\phi_2}{\partial u}\right)\left(D_1'\frac{\partial\phi_1}{\partial v} - D_1''\frac{\partial\phi_1}{\partial u}\right)}{2F_1}$$

Hence, if any two surfaces are given which correspond by orthogonality of linear elements and a third surface can be found having the same kind of correspondence with either, and which is such that the characteristic functions of the latter satisfy these equations, the two original surfaces can be deformed in such a

way that the resulting surfaces will have the same kind of correspondence. This condition is satisfied if ϕ_2 is such that

$$D_1 \frac{\partial \phi_2}{\partial v} - D_1' \frac{\partial \phi_2}{\partial u} = 0, \quad D_1' \frac{\partial \phi_2}{\partial v} - D_1'' \frac{\partial \phi_2}{\partial u} = 0,$$

that is, excluding the case where S_1 is a developable surface,

$$\phi_2 = \text{const.}$$

As we have seen, this holds when S_1 and S_2 are adjoint minimal surfaces. Hence, if S_1 is a minimal surface and is deformed infinitesimally along directrices whose direction cosines are proportional to the point coordinates of its adjoint, the resulting surface corresponds by orthogonality of linear elements to the surface S'.

Suppose that for S_2 we take the surface S, then $\phi_1 = \phi_2$ and the ratios (9) reduce to

$$\frac{M_1^2}{E_1} = \frac{N_1^2}{G_1} = \frac{M_1 N_1}{F_1},$$

which has for common value zero. Hence, neglecting the case where S_1 is a developable surface, we have

$$\phi_1 = \text{const.};$$

that is, S and S_1 are minimal surfaces and S_1 is the adjoint of S or a homothetic surface of this adjoint. It follows, therefore, that the only case where the non-developable surfaces S and S_1 are such that the surfaces resulting from an infinitesimal deformation of them, defined by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \delta z_1,$$

and

$$x_1' = x_1 + \varepsilon_1 x, \quad y_1' = y_1 + \varepsilon_1 y, \quad z_1' = z_1 + \varepsilon_1 z,$$

correspond by orthogonality of linear elements, is that in which S is a minimal surface and S_1 its adjoint or a homothetic of the latter. In particular, if S and S_1 are adjoint minimal surfaces, the respective deformations giving this result are defined by

$$x' = x + \varepsilon x_1, \quad y' = y + \varepsilon y_1, \quad z' = z + \varepsilon z_1,$$

and

$$x_1' = x_1 - \varepsilon x, \quad y_1' = y_1 - \varepsilon y, \quad z_1' = z_1 - \varepsilon z.$$

Another case is that in which the surfaces resulting from an infinitesimal deformation of S and S_0 correspond by parallelism of tangent planes. For this

discussion we make use of the following expressions for the direction cosines of the tangent planes to S' found by Genty:*

$$X' = X + \varepsilon (Y z_0 - Z y_0),$$
$$Y' = Y + \varepsilon (Z x_0 - X z_0),$$
$$Z' = Z + \varepsilon (X y_0 - Y x_0).$$

If the associate surface of S_0 in the given deformation is denoted by $\Sigma(\xi, \eta, \zeta)$, we have

$$X_0' = X_0 + \varepsilon_1 (Y_0 \zeta - Z_0 \eta),$$

and similar expressions for Y_0' and Z_0'. But

$$X_0 = X, \quad Y_0 = Y, \quad Z_0 = Z,$$

hence, if the given correspondence holds between S' and S_0', it is necessary that

$$\frac{x_0 - \tau\xi}{X_0} = \frac{y_0 - \tau\eta}{Y_0} = \frac{z_0 - \tau\zeta}{Z^0}, \tag{10}$$

where $\tau = \frac{\varepsilon_1}{\varepsilon}$.

When, in particular, we take $\varepsilon_1 = \varepsilon$, these equations become

$$\frac{x_0 - \xi}{X_0} = \frac{y_0 - \eta}{Y_0} = \frac{z_0 - \zeta}{Z_0}. \tag{11}$$

Since S_0 and Σ correspond by parallelism of tangent planes, it follows from (11) that they are parallel surfaces. Comparing equations (10) and (11), we remark that when $\varepsilon \neq \varepsilon_1$, Σ is a homothetic of a surface parallel to S_0. Since the converse of the above results is true, we have the following theorem:

The necessary and sufficient condition that S_0', arising from an infinitesimal deformation of S_0, corresponds to S' by parallelism of tangent planes at corresponding points is that Σ be parallel to S_0, or a homothetic of a parallel.

Let Σ be a parallel of S_0 and denote by D_λ, D_λ', D_λ'' the coefficients of its second fundamental form. From (III, 11) we find, on remarking that S' and S_0', have the same spherical representation, and that for any surface $HK = \mathcal{K}$,

$$\frac{D_0 - D_\lambda}{\mathfrak{E}_0} = \frac{D_0' - D_\lambda'}{\mathcal{F}_0} = \frac{D_0'' - D_\lambda''}{\mathcal{G}_0}. \tag{12}$$

* Toulouse Annales, t. IX, E. 7.

The second fundamental coefficients of parallel surfaces are given by*

$$D_p = E_0 \frac{a}{\rho_1\rho_2} + D_0\left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$
$$D'_p = F_0 \frac{a}{\rho_1\rho_2} + D'_0\left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$
$$D''_p = G_0 \frac{a}{\rho_1\rho_2} + D''_0\left[1 - a\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right],$$

where ρ_1 and ρ_2 are the principal radii of normal curvature of S_0. Substituting these values in (12), we have

$$\frac{E_0 - D_0(\rho_1 + \rho_2)}{\mathfrak{E}_0} = \frac{F_0 - D'_0(\rho_1 + \rho_2)}{\mathcal{F}_0} = \frac{G_0 - D''_0(\rho_1 + \rho_2)}{\mathcal{G}_0},$$

By making use of (11), we find that the common value of these ratios is $-\rho_1\rho_2$. Since these expressions involve the coefficients of only one surface, and since this surface is perfectly general, we have for any surface

$$E = D\ (\rho_1 + \rho_2) - \mathfrak{E}\rho_1\rho_2,$$
$$F = D'\ (\rho_1 + \rho_2) - \mathcal{F}\rho_1\rho_2,$$
$$G = D''(\rho_1 + \rho_2) - \mathcal{G}\rho_1\rho_2;$$

these reduce to formulæ (3), when S is a minimal surface.

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* Knoblauch, "Allgemeine Theorie der Krummen Flächen," p. 236.